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AN IMPROVED ALGEBRAIC RELATION FOR THE CALCULATION OF REYNOLDS STRESSES

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1. Introduction

The prediction capability of a turbulence model depends on how effectively one can prescribe the Reynolds stress distribution in closing the system of equations. The simplest and most widely used has been the Boussinesq treatment of the Reynolds stresses. As is now well known, Boussinesq hypothesis holds only when the strain rates are fairly small. The main reason being that by construction the Boussinesq formula implies that the principal axes of the Reynolds stress tensor are parallel to the principal axes of the strain rate tensor so that any change in the strain rate is directly felt in the stresses. This instantaneous change of the Reynolds stresses with the strain rates is not supported by the experimental observations, because the Reynolds stresses being due to the vorticity fluctuations require some time to adjust to the new strain rates. To overcome these short comings, one must either abandon the Boussinesq hypothesis altogether and solve the six Reynolds transport equations which is costly in terms of computer time or improve upon the hypothesis itself.

In this paper we follow a recent analysis of Rodi[1] to construct an improved second-order version of the Boussinesq hypothesis. An algebraic relation for the turbulent stresses has been obtained through a consideration of the transport equations of the Reynolds stresses. Consequently, the resulting relation has the necessary influence of the convective and diffusive transport effects of a turbulence stress field.

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In this paper an algebraic relation for the Reynolds Stresses has been obtained through a consideration of the transport equations of the Reynolds Stresses. This analysis provides a second-order approximation to the Boussinesq eddy viscosity hypothesis. The basic assumption of the analysis is that the derivatives of the ratio of the Reynolds Stresses and energy is small in comparison with the other terms.

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2. Analysis

The transport equations of the Reynolds stresses $(\overline{-u_i u_j})$ and the equation of turbulence energy $(\bar{e} = \frac{1}{2}\overline{u_i u_i})$ for an incompressible flow respectively are

$$\frac{d\tau_{ij}}{dt} = P_{ij} + Q_{ij} + D_{ij} - \varepsilon_{ij}\,, \qquad \tau_{ij} = \overline{u_i u_j} \tag{1}$$

$$\frac{d\bar{e}}{dt} = P + D - \varepsilon \tag{2}$$

where $\frac{d}{dt}$ is the substantive derivative based on the mean velocity components U_i; P_{ij}, D_{ij}, ε_{ij} respectively are the production, diffusion and dissipation of the Reynolds stresses, Q_{ij} is the pressure-strain correlation, while P, D and ε respectively are the production, diffusion and dissipation of the turbulence energy. In this paper we have utilized the modeling of the terms Q_{ij}, D_{ij}, ε_{ij} and D as reported in references 1 and 2, which on using the summation convention on repeated indices are

$$Q_{ij} = \frac{c_1 \varepsilon}{\bar{e}}\left(\frac{2}{3}\bar{e}\,\delta_{ij} - \tau_{ij}\right) + \gamma\left(\frac{2P}{3}\delta_{ij} - P_{ij}\right) \tag{3}$$

$$D_{ij} = \frac{\partial}{\partial x_k}\left(\nu\frac{\partial \tau_{ij}}{\partial x_k} + \frac{c_s \bar{e}}{\varepsilon}\tau_{k\ell}\frac{\partial \tau_{ij}}{\partial x_\ell}\right) \tag{4}$$

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\,\delta_{ij} \tag{5}$$

$$D = \frac{\partial}{\partial x_k}\left(\nu\frac{\partial \bar{e}}{\partial x_k} + \frac{c_s \bar{e}}{\varepsilon}\tau_{k\ell}\frac{\partial \bar{e}}{\partial x_\ell}\right) \tag{6}$$

where c_1, γ and c_s are empirical constants, and ν the kinematic viscosity. The terms P_{ij}, p and ε are

$$P_{ij} = -\left(\tau_{ik}\frac{\partial U_j}{\partial x_k} + \tau_{jk}\frac{\partial U_i}{\partial x_k}\right) \tag{7}$$

$$P = \frac{1}{2}P_{ii} = -\tau_{k\ell}\frac{\partial U_i}{\partial x_\ell} \tag{8}$$

$$\varepsilon = \nu\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} \tag{9}$$

Introducing the notation

$$T_{ij} = \tau_{ij}/\bar{e} \tag{10}$$

and arranging terms in D_{ij}, we have

$$D_{ij} = \nu \frac{\partial}{\partial x_k} (\bar{e} \frac{\partial T_{ij}}{\partial x_k}) + \nu \frac{\partial \bar{e}}{\partial x_k} \frac{\partial T_{ij}}{\partial x_k} + DT_{ij}$$

$$+ \frac{\partial}{\partial x_k} \left\{ \frac{c_s (\bar{e})^2}{\varepsilon} \tau_{k\ell} \frac{\partial T_{ij}}{\partial x_\ell} \right\} + \frac{c_s \bar{e}}{\varepsilon} \tau_{k\ell} \frac{\partial \bar{e}}{\partial x_\ell} \frac{\partial T_{ij}}{\partial x_k} \tag{11}$$

Introducing (11), the identity

$$\frac{d\tau_{ij}}{dt} = \bar{e} \frac{dT_{ij}}{dt} + T_{ij} \frac{d\bar{e}}{dt}$$

and Eq. (2) in Eq. (1) and neglecting the derivatives of T_{ij} in comparison with the other terms, we obtain

$$T_{ij}(P-\varepsilon) = P_{ij} + Q_{ij} - \varepsilon_{ij} \tag{12}$$

On substituting (3) and (5) in (12) we obtain

$$T_{ij} = \frac{2}{3} \delta_{ij} + \gamma_o (P_{ij}/\varepsilon - \frac{2}{3} \delta_{ij} P/\varepsilon)/(d_1 + P/\varepsilon) \tag{13}$$

where $\gamma_o = 1 - \gamma$ and $d_1 = c_1 - 1$

We now introduce the following notation

$$\theta^2 = \frac{1}{2}\overline{\omega_i \omega_i}, \quad M_{ij} = \frac{1}{\theta} \frac{\partial U_i}{\partial x_j} \tag{14}$$

where θ is the vorticity-density and ω_i is the fluctuating vorticity component. It follows directly from the Kolmogorov-Saffman equation of energy (Ref. 3) that the dissipation of energy ε is given by

$$\varepsilon = \bar{e}\theta \tag{15}$$

Using (14) and (15) in (7) and (8) we get

$$P_{ij}/\varepsilon = -(T_{ik} M_{jk} + T_{jk} M_{ik}) \tag{16}$$

$$P/\varepsilon = -T_{ki} M_{ki} \tag{17}$$

If we now substitute (16) and (17) in (13) then we get a system of nonlinear simultaneous algebraic equations for the determination of T_{ij}. Rodi[1] in his derivation did not use the expansion (17), but retained P/ε as a parameter and solved (13) for T_{ij}. Since P/ε contains all T_{ij}'s, we follow an approach different from Rodi, which in the first place establishes the validity of the Boussinesq hypothesis, and in the second place yields an improved algebraic relation for T_{ij}.

Equation (13) can be written in various iterative forms, however, the following form is chosen because it yields various approximations in a direct fashion.

$$d_1 T_{ij}^{(n+1)} - T_{ij}^{(n)} T_{k\ell}^{(n)} M_{k\ell}$$
$$= \frac{2}{3}\delta_{ij}(d_1 - T_{k\ell}^{(n)} M_{k\ell}) + \gamma_o(-T_{ik}^{(n)} M_{jk} - T_{jk}^{(n)} M_{ik} + \frac{2}{3}\delta_{ij} T_{k\ell}^{(n)} M_{k\ell}) \quad (18)$$

where n is the iteration index.

For the zeroeth approximation we take

$$T_{ij}^{(0)} = \frac{2}{3}\delta_{ij}$$

and by using the continuity equation $M_{jj} = 0$, we get

$$T_{ij}^{(1)} = \frac{2}{3}\delta_{ij} - \alpha_o^2(M_{ij} + M_{ji}) \quad (19)$$

where

$$\alpha_o^2 = \frac{2\gamma_o}{3d_1} = \text{constant}. \quad (20)$$

Equation (19) is the usual Boussinesq formulation which has also been used among others by Kolmogorov[4] and Saffman[3]. It is only a first approximation and is expected to hold in situations where the mean flow is not changing rapidly.

To obtain the second approixmation, we introduce (19) in (18) and neglect terms of the third-order in M_{ij} to have

$$T_{ij}^{(2)} = \frac{2}{3}\delta_{ij} - \alpha_o^2(M_{ij} + M_{ji})$$
$$+ \frac{3}{2}\alpha_o^4[(M_{ik} + M_{ki})M_{jk} + (M_{jk} + M_{kj})M_{ik} - \frac{2}{3}\delta_{ij}(M_{k\ell} + M_{\ell k})M_{k\ell}] \quad (21)$$

Equation (21) provides the second approximation to the Reynolds stresses and is expected to hold from low to moderate variations of the strain rates.

By following a philosophically different approach Saffman[3] has also obtained an expression similar to (21) which in our notation is

$$T_{ij}^{(2)} = \frac{2}{3}\delta_{ij} - \alpha^2(M_{ij} + M_{ji}) + $$
$$+ \frac{\lambda}{2}[(M_{ik} + M_{ki}) M_{jk} + (M_{jk} + M_{kj})M_{ik}$$
$$- \left\{(M_{jk} + M_{kj})M_{ki} + (M_{ki} + M_{ik})M_{kj}\right\}] \quad (22)$$

where α and λ are constants. Comparing (21) and (22) we find that though both are in dimensional agreement, they differ in their last terms. It must be noted that (21) is a consequence of the complete Navier-Stokes equations while (22) is an attempt at finding a relaxation model to overcome the difficiencies of the first approximation.

A comparison of (21) and (22) yields the values of the constant α_o appearing in (21). Thus we have

$$\alpha_o = \alpha = 0.3$$

The value $\alpha_o = 0.3$ has been used by Saffman[3] and also by Pope and Whitelaw[5], but from (20) we find that the value $\alpha_o = 0.3$ is not consistent with the values $\gamma_o = 0.4$ and $d_1 = 0.5$ as proposed in Ref. 2. However, if we take the value $d_1 = 1.86 \simeq (\frac{2}{.7} - 1)$ as mentioned by Rotta[6] and $\gamma_o = 0.3$ then $\alpha_o \simeq 0.33$, which is near to the value used by Saffman. For the sake of definiteness we therefore select the value $\alpha_o = 0.3$ in Eq. (21).

For two-dimensional mean flow Eq. (21) yields the following expressions for the normal and shear stresses:

$$T_{11}^{(2)} = T_{11}^{(1)} + \alpha_o^4[2M_{11}^2 + 3(M_{12} + M_{21})M_{12} - (M_{12} + M_{21})^2] \quad (23a)$$

$$T_{22}^{(2)} = T_{22}^{(1)} + \alpha_o^4[2M_{11}^2 + 3(M_{12} + M_{21})M_{21} - (M_{12} + M_{21})^2] \quad (23b)$$

$$T_{33}^{(2)} = T_{33}^{(0)} - \alpha_o^4[4M_{11}^2 + (M_{12} + M_{21})^2] \quad (23c)$$

$$T_{12}^{(2)} = T_{12}^{(1)} - 3\alpha_o^4 M_{11}(M_{12} - M_{21}) \quad (23d)$$

On the other hand Saffman's equations (22) yields

$$T_{11}^{(2)} = T_{11}^{(1)} + \lambda(M_{12} + M_{21})(M_{12} - M_{21}) \tag{24a}$$

$$T_{22}^{(2)} = T_{22}^{(1)} - \lambda(M_{12} + M_{21})(M_{12} - M_{21}) \tag{24b}$$

$$T_{33}^{(2)} = T_{33}^{(0)} \tag{24c}$$

$$T_{12}^{(2)} = T_{12}^{(1)} - 2\lambda M_{11}(M_{12} - M_{21}) \tag{24d}$$

where λ is an emperical constant. Thus though the shear stresses, Eqs. (23d) and (24d), have the same distributions, the normal stresses, Eqs. (23a-c) and Eqs. (24a-c), have entirely different distributions.

In the wall region $M_{11} = M_{22} = M_{21} \simeq 0$ and

$$M_{12} \simeq \frac{1}{\alpha_o}$$

so that Eqs. (23) become

$$T_{11}^{(2)} - \frac{2}{3} = 2\alpha_o^2 \tag{25a}$$

$$T_{22}^{(2)} - \frac{2}{3} = -\alpha_o^2 \tag{25b}$$

$$T_{33}^{(2)} - \frac{2}{3} = -\alpha_o^2 \tag{25c}$$

$$T_{12}^{(2)} = -\alpha_o \tag{25d}$$

Equations (24) become

$$T_{11}^{(2)} - \frac{2}{3} = \lambda/\alpha_o^2 \tag{26a}$$

$$T_{22}^{(2)} - \frac{2}{3} = -\lambda/\alpha_o^2 \tag{26b}$$

$$T_{33}^{(2)} = \frac{2}{3} \tag{26c}$$

$$T_{12}^{(2)} = -\alpha_o \tag{26d}$$

Saffman[3] now takes $\lambda = .02$ to match the three normal stresses with the experimental data which roughly are in the ratio 4:2:3. Numerical values based on (25) and (26) and the experimental values as quoted in Ref. 7 are tabulated below.

Table 1 Comparison of the Near-Wall Data

	Present(α=0.3)	Present(α=0.34)	Saffman[3]	Reference 7
$T_{11}^{(2)} - \frac{2}{3}$	0.18	0.23	0.22	0.32
$T_{22}^{(2)} - \frac{2}{3}$	-0.09	-0.12	-0.22	-0.18
$T_{33}^{(2)} - \frac{2}{3}$	-0.09	-0.12	0.0	-0.10
$T_{12}^{(2)}$	-0.30	-0.34	-0.30	-0.34

A comparison of values in Table 1 shows that α_o=0.34 in the present model may be more suitable than α_o=0.3. However, any adjustment of the constant α_o or the actual prediction capability of the proposed second-order algebraic relation (21) can be ascertained only after it has been used in the calculation of various turbulent flows.

Based on the works of Launder et al[2] and on the most recent review by Reynolds[8] it is possible to establish in advance the limitations of the second approximation, viz. Eq. (21). For example, it may be observed that in the wall region the normal stresses $T_{22}^{(2)}$ and $T_{33}^{(2)}$ are equal (Eqs. 25b,c). This result is in exact conformity with Eqs. (14) of Ref. 2 in which Launder et al have shown that in any simpler pressure-strain hypothesis there is no direct production of T_{22} and T_{33} and therefore they tend to be equal. Thus it can roughly be stated that the amount of approximation involved in (21) is the same as obtaining the Reynolds stresses through the six Reynolds stress transport equations by using a simpler pressure-strain relation.

3. Conclusion

The purpose of this paper has been to demonstrate that the Boussinesq eddy viscosity hypothesis and its higher approximations are a direct consequence of the Reynolds stress transport equations. The basic assumption of the analysis is that the derivatives of the terms $T_{ij} = \frac{\tau_{ij}}{\bar{e}}$ are small in comparison with the other terms in the rate equation for τ_{ij}. Because of the implicit effects of the convection and diffusion in the second approximation, the simple algebraic relation (21) is expected to provide a basis for the prediction of complex flows.

It is important to mention that the validity of Eq. (13), which rests on the assumption that the rate of change of T_{ij} be small in comparison with the other terms is not new either in the paper by Rodi[1] or in the present one. This idea [§] has earlier been used by Donaldson[9] who called it as the "super-equilibrium" limit. As an application, Donaldson obtained all the Reynolds stress terms algebraically for a line vortex.

[§] The authors are grateful to the reviewer for bringing this to our attention.

REFERENCES

1. Rodi, W., "A New Algebraic Relation for Calculating the Reynolds Stresses," ZAMM, Vol. 56, 1976, pp. T219-T221.

2. Launder, B. E., Reece, G. J., and Rodi, W., "Progress in the Development of a Reynolds Stress Closure," J. Fluid Mechanics, Vol. 68, 1975, pp. 537-566.

3. Saffman, P. G., "Model Equations for Turbulent Shear Flow," Studies in Applied Math., Vol. 53, 1974, pp. 17-34.

4. Kolmogorov, A. N., "Equations of Turbulent Motion of an Incompressible Fluid," Izv. Akad. Nauk SSR Seria Fizichiska, Vol. 6, p. 56.

5. Pope, S. B. and Whitelaw, J. H. "The Calculation of Near-Wake Flows," J. Fluid Mech., Vol. 73, 1976, pp. 9-32.

6. Rotta, J. C., "Turbulent Boundary Layers in Incompressible Flow," Progress in Aeronautical Sciences, Vol. 2, Pergamon Press, 1962, p.48.

7. Mellor, G. L. and Herring, H. J., "A Survey of the Mean Turbulent Field Closure Models," AIAA J., Vol. 11, 1973, pp. 590-599.

8. Reynolds, W. C., "Computation of Turbulent Flows," Annual Review of Fluid Mechanics, Vol. 8, 1976, pp. 183-208.

9. Donaldson, C., dup., "The Relationship Between Eddy-Transport and Second-Order Closure Models for Stratified Media and for Vortices," Proc. NASA Conference on Free Turbulent Shear Flows, July 20-21, 1972. NASA Publ. No. SP-321, pp. 233-258.